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LETTER TO THE EDITOR

The unitary supermultiplet of $N=8$ conformal superalgebra involving fields of spin $\leqslant 2$†

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Abstract. We construct the unique unitary supermultiplet of $N=8$ conformal superalgebra SU(2, 2/8) in $d=4$ which involves fields of spin $s \leqslant 2$. There exists a one-to-one correspondence between the fields of the $N=8$ conformal supermultiplet and those of the $N=8$ Poincaré supergravity. The $N=4$ analogue of this supermultiplet is that of $N=4$ Yang–Mills theory which is conformally invariant. If this $N=8$ conformal supermultiplet can be used to construct a super-conformally invariant theory then we expect the resulting theory to be different from the standard conformal supergravity theories.

In previous work on conformal supergravity it has been argued that there cannot be a physically acceptable conformal supergravity theory with more than four supersymmetries ($N>4$). Such theories are thought to be plagued with ghosts and involve fields of spin greater than two [1, 2]. In this letter we show that there exists a *unitary* supermultiplet of the $N=8$ super-conformal algebra SU(2, 2/8) involving fields of *spin less than or equal to two.* This conformal supermultiplet differs in several respects from the standard conformal supermultiplets used in constructing conformal supergravity theories. For example the vector fields in this supermultiplet do not transform as the adjoint representation of the internal symmetry group U(8). This supermultiplet is analogous to the singleton supermultiplet of $N=8$ anti-de Sitter supergroup in $d=4$ [3] and the doubleton multiplets of the N-extended anti-de Sitter supergroups in odd-dimensional spacetimes [4, 5]. In fact if we interpret the conformal group SO(4, 2) as the anti-de Sitter group in $d=5$ then it is precisely the self-conjugate doubleton supermultiplet of $N=16$, $d=5$ anti-de Sitter superalgebra. Therefore we shall refer to this supermultiplet as the self-conjugate $N=8$ conformal doubleton multiplet or simply as the $N=8$ conformal doubleton. Remarkably enough, the fields of the $N=8$ conformal doubleton supermultiplet are in a one-to-one correspondence with the fields of the $N=8$ Poincaré supergravity [6]. Therefore if there exists a conformal supergravity theory whose physical fields are those of the $N=8$ conformal doubleton then we expect the 70 scalars of such a theory to parametrise the coset space $E_{7(7)}/\mathrm{SU}(8)$ as in the $N=8$ Poincaré supergravity [6]. A strong indication for the existence of such a theory is provided by the fact that there exists a conformally invariant theory whose physical fields are those of the self-conjugate $N=4$ conformal doubleton supermultiplet, namely the $N=4$ Yang–Mills theory [4, 7].

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Before constructing the $N=8$ conformal doubleton supermultiplet let us recall certain basic facts about N-extended conformal superalgebras. The N-extended conformal superalgebra $SU(2,2/N)$ has as its even subalgebra $SU(2,2)\times U(N)$ where $U(N)=SU(N)\times U(1)_{\text{Axial}}$ and $SU(2,2)=SO(4,2)$ is the Lie algebra of the conformal group in four dimensions. The generators of $SO(4,2)$ are the usual Poincaré group generators $J_{\mu\nu}$, P_μ plus the generators of special conformal transformations K_μ and the dilatations D. The odd generators of $SU(2,2/N)$ are the N supersymmetry generators Q^i_a $(i=1,\ldots,N)$ and the N special supersymmetry generators S^i_a. For the full set of commutation and anticommutation relations of $SU(2,2/N)$ we refer to reference [8]. Our aim here is to construct the unitary irreducible representation of $SU(2,2/8)$ corresponding to the $N=8$ conformal doubleton supermultiplet. For the construction of the unitary representations of $SU(2,2/8)$ we shall use the oscillator method of references [9-11]. This method gives all the lowest weight (or equivalently all the positive 'energy') unitary irreducible representations of non-compact groups $SU(n,m)$, $SO^*(2n)$ and $Sp(2n,\mathbb{R})$ and of non-compact supergroups $Osp(2n/2m,\mathbb{R})$, $Osp(2n^*/2m)$ and $SU(n,m/p)$†. In this method the generators of the non-compact supergroup are realised as bilinears of bosonic and fermionic oscillators.

The general construction of the oscillator-like (lowest-weight) unitary representations of $SU(n,m/p)$ was studied in detail in reference [10]. Here we shall specialise to $SU(2,2/8)$ and consider those unitary representations that can be obtained when one realises the Lie superalgebra of $SU(2,2/8)$ as bilinears of two sets of Bose–Fermi oscillators. Hence these representations will be generically referred to as the doubleton representations. We should note that to generate all the oscillator-like (lowest-weight) unitary representations one has to consider arbitrarily many pairs of Bose–Fermi oscillators.

For the construction of the unitary irreducible representations we shall decompose the generators of $SU(2,2/8)$ as [10]

$$L=L^-\oplus L^0\oplus L^+, \tag{1}$$

where L^0 represents the generators of the maximal compact subsupergroup $SU(2/4)\times SU(2/4)\times U(1)$, where $SU(2/4)$ is a compact supergroup with even subgroup $S(U(2)\times U(4))$. Equation (1) corresponds to the Jordan decomposition (three grading) of $SU(2,2/8)$ [10]. We shall take two sets of Bose–Fermi oscillators $\xi_A(\xi^A)$ and $\eta_M(\eta^M)$ where the A and M are the indices of the two different $SU(2/4)$ sub-supergroups. We shall denote annihilation operators with lower indices and creation operators with upper indices. The first two components of ξ and η are bosonic oscillators and the last four components are fermionic, i.e.,

$$\begin{aligned}
&\xi_A=\begin{pmatrix}a_i\\ \alpha_\mu\end{pmatrix} \qquad \xi^A=\begin{pmatrix}a^i\\ \alpha^\mu\end{pmatrix} \qquad \begin{array}{l} i,j\ldots=1,2\\ \mu,\nu\ldots=1,\ldots,4\end{array}\\
&\eta_M=\begin{pmatrix}b_r\\ \beta_x\end{pmatrix} \qquad \eta^M=\begin{pmatrix}b^r\\ \beta^x\end{pmatrix} \qquad \begin{array}{l} r,s\ldots=1,2\\ x,y\ldots=1,\ldots,4.\end{array}
\end{aligned}\tag{2}$$

They satisfy the canonical super-commutation relations

$$\begin{aligned}
&\{\xi_A,\xi^B]=\delta_A{}^B\\
&\{\eta_M,\eta^N]=\delta_M{}^N,
\end{aligned}\tag{3}$$

† For a discussion of this point see [5].

where { ,] denotes an anti-commutator for two fermionic oscillators and a commutator otherwise. The generators of the two SU(2/4) are

$$I_A{}^B = \xi_A \xi^B + \tfrac{1}{2}\delta_A{}^B(\xi_C \xi^C), \qquad J_M{}^N = \eta_M \eta^N + \tfrac{1}{2}\delta_M{}^N(\eta_Q \eta^Q). \tag{4}$$

The U(1) generator in L^0 can be taken as the number operator $N = \xi^C \xi_C + \eta^Q \eta_Q$ of all the oscillators. The 'non-compact' generators belonging to L^+ and L^- spaces are

$$L_{AM} = \xi_A \eta_M \in L^-, \qquad L^{MA} = L^\dagger_{AM} = \eta^M \xi^A \in L^+. \tag{5}$$

To construct a UIR of SU(2, 2/8) one starts from a ground state $|\Omega\rangle$ which transforms irreducibly under L^0 and is annihilated by the generators belonging to L^-. Then by acting on $|\Omega\rangle$ with L^+ repeatedly one generates an infinite set of states which form a UIR of SU(2, 2/8):

$$\mathbb{R} = \{|\Omega\rangle, L^+|\Omega\rangle, L^{+2}|\Omega\rangle, \ldots\}. \tag{6}$$

For $N = 8$ the only ground state $|\Omega\rangle$ that leads to a unitary conformal supermultiplet containing no fields of spin greater than two is the true Fock vacuum $|0\rangle$ i.e. the state annihilated by all the annihilation operators. The resulting supermultiplet is the shortest non-trivial self-conjugate supermultiplet of $N = 8$ conformal superalgebra and is analogous to the singleton representation of the $N = 8$ anti-de Sitter superalgebra in $d = 4$ [3]. The infinite set of states obtained by successive applications of the L^+ generators $\xi^A \eta^M$ on the vacuum

$$|0\rangle, L^+|0\rangle = \xi^A \eta^M |0\rangle, (L^+)^2|0\rangle, \ldots \tag{7}$$

that form the basis of a UIR of SU(2, 2/8) can be decomposed into unitary irreducible representations of the even subgroup SU(2, 2) × U(8). The unitary irreducible representations of SU(2, 2) are all infinite dimensional and the class of unitary representations of SU(2, 2) are all infinite dimensional and the class of unitary representations one obtains by the oscillator method are all of the lowest weight type or equivalently of the positive 'energy' type. The 'energy' operator is defined to be the generator E of the Abelian U(1) of the maximal compact subgroup $SU(2)_{j_1} \times SU(2)_{j_2} \times U(1)_E \approx SO(4) \times SO(2)_E$. This 'conformal energy' operator is simply $E = P^0 = K^0$ where P^0 and K^0 are the time components of momentum and special conformal generators. (If one considers SO(4, 2) as the anti-de Sitter energy operator.) The Lie algebra B of SU(2, 2) has a Jordan decomposition (three grading) determined by the 'conformal energy' operator E [9]

$$B = B^- \oplus B^0 \oplus B^+, \tag{8}$$

where B^0 is the Lie algebra of the $SU(2)_{j_1} \times SU(2)_{j_2} \times U(1)_E$ subgroup. The generators in B^0 are $I_i{}^j$, $J_r{}^s$ and $E = \tfrac{1}{2}(a^i a_i + b_r b^r)$. The generators belonging to B^- and B^+ spaces are $a_i b_r$ and $a^j b^s$, respectively.

A positive 'energy' (or lowest weight) UIR of SU(2, 2) is *uniquely* determined by a lowest state $|w\rangle$ that is annihilated by all the operators in B^- space and that transforms *irreducibly* under the maximal component subgroup $SU(2)_{j_1} \times SU(2)_{j_2} \times U(1)_E$ generated by B^0 [9]. Starting from such a state $|w\rangle$ one generates an infinite set of states by repeated application of the B^+ operators

$$|w\rangle, B^+|w\rangle, (B^+)^2|w\rangle, \ldots, \tag{9}$$

which form the basis of an UIR of SU(2, 2). Considered as the anti-de Sitter (AdS) group in $d = 5$ the infinite set of states (9) in a given UIR of SO(4, 2) can be identified with the positive energy Fourier modes of a field in AdS_5 [4]. This field is uniquely determined by the lowest state $|w\rangle$. Similarly in $d=4$ we shall identify the infinite set of states with the positive 'conformal energy' Fourier modes of a field.

As for the compact internal symmetry group SU(8) all the unitary representations are finite dimensional and are of the lowest weight type (or equivalently of the highest weight type). The states with definite spacetime transformation properties under $SU(2)_{j_1}\times SU(2)_{j_2}\times U(1)_E$ in a given UIR of SU(2, 2/8) fall into irreducible representations of SU(8). To determine these irreps we use the Jordan decomposition of SU(8) with respect to its subgroup $SU(4)\times SU(4)\times U(1)$. In terms of the fermion bilinears the Lie algebra F of SU(8) has the three graded structure

$$F=F^{-}\oplus F^{0}\oplus F^{+}, \tag{10}$$

where F^0 contains the $SU(4)\times SU(4)\times U(1)$ generators $I_\mu{}^\nu$, $J_x{}^y$ and $Q=\alpha^\mu\alpha_\mu-\beta_x\beta^x$ and F^- and F^+ spaces correspond to the generators $L_{\mu x}=\alpha_\mu\beta_x$ and $L^{x\mu}=\beta^x\alpha^\mu$, respectively. Again the irreps of SU(8) that occur in (7) are uniquely determined by their lowest state $|\nu\rangle$ which transforms irreducibily under $F^0=SU(4)\times SU(4)\times U(1)$ and is annihilated by F^-. Starting from such a state one generates an irrep of SU(8) by repeated applications of F^+ operators

$$|\nu\rangle,\ F^+|\nu\rangle,\ (F^+)^2|\nu\rangle,\ \ldots. \tag{11}$$

Since F^+ consists of fermion bilinears the series (11) terminates after a finite number of steps and the resulting unitary representation is finite dimensional. Thus to decompose the infinite set of states in the UIR of SU(2, 2/8) given in (7) we need to determine those states that are annihilated by both the B^- operators L_{ir} and the F^- operators $L_{\mu x}$ and that transforms irreducibly under B^0 and F^0. Such states will be called the *BF* lowest states. In table 1 we list the *BF* lowest states that occur in the UIR obtained by choosing as the ground state the true Fock vacuum $|0\rangle$. By acting on the *BF* lowest states with the operators in B^+ and F^+ repeatedly one generates unitary irreducible representations of SU(2, 2) transforming in a definite irrep of SU(8). The second and third columns in table 1 give the transformation properties of the *BF* lowest states under the maximal compact subgroup $SU(2)_{j_1}\times SU(2)_{j_2}\times U(1)_E$ of the conformal group SU(2, 2), which determine uniquely the resulting UIR of SU(2, 2). The fourth column in table 1 lists the SU(8) transformation properties of these UIRs of SU(2, 2), whereas the fifth column gives their axial $U(1)_A$ quantum numbers. The last column gives the corresponding fields. The indices $i, j, \ldots=1,\ldots,8$ on the fields refer to SU(8) indices whereas the indices $\alpha, \beta, \ldots=1, 2$ and $\dot\alpha, \dot\beta, \ldots$ refer to the $SL(2, C)$ spinor indices

Table 1. The self-conjugate $N=8$ conformal doubleton supermultiplet.

BF lowest states	$SU(2)_{j_1}\times SU(2)_{j_2}$	E_0	SU(8)	$U(1)_A$	Fields
$\lvert 0\rangle$	$(0,0)$	1	70	0	$\varphi^{[ijkl]}$
$L^{ix}\lvert 0\rangle$	$(\frac{1}{2},0)$	$\frac{3}{2}$	56	1	$\lambda_+^{[ijk]}\equiv\lambda_a^{[ijk]}$
$L^{r\mu}\lvert 0\rangle$	$(0,\frac{1}{2})$	$\frac{3}{2}$	$\overline{56}$	-1	$\lambda_{-[ijk]}\equiv\lambda_{a[ijk]}$
$(L^{ix})^2\lvert 0\rangle$	$(1,0)$	2	28	2	$F_{\mu\nu}^{+[ij]}\equiv F_{(\alpha\beta)}^{[ij]}$
$(L^{r\mu})^2\lvert 0\rangle$	$(0,1)$	2	$\overline{28}$	-2	$F^-_{\mu\nu[ij]}\equiv F_{(\dot\alpha\dot\beta)[ij]}$
$(L^{ix})^3\lvert 0\rangle$	$(\frac{3}{2},0)$	$\frac{5}{2}$	8	3	$\psi_\mu^{+i}\equiv\psi^i_{(\alpha\beta\gamma)}$
$(L^{r\mu})^3\lvert 0\rangle$	$(0,\frac{3}{2})$	$\frac{5}{2}$	$\bar{8}$	-3	$\psi^-_{\mu i}\equiv\psi_{(\dot\alpha\dot\beta\dot\gamma)i}$
$(L^{ix})^4\lvert 0\rangle$	$(2,0)$	3	1	4	$R_{(\alpha\beta\gamma\delta)}$
$(L^{r\mu})^4\lvert 0\rangle$	$(0,2)$	3	1	-4	$R_{(\dot\alpha\dot\beta\dot\gamma\dot\delta)}$

and their conjugates, respectively. The square brackets [ ] on the indices refer to anti-symmetrisations and the round brackets ( ) to symmetrisations. It is clear from this table that one can establish a one-to-one correspondence between the self-conjugate $N=8$ conformal doubleton multiplet and the self-conjugate $N=8$ Poincaré supermultiplet. However we should note that some of the fields appear through their field strengths in the conformal supermultiplet. For example the 28 vector fields appear not as the $(\frac{1}{2},\frac{1}{2})$ representation of the Lorentz group $SL(2,\mathbb{C})$ but rather as the $(1,0)+(0,1)$ representation corresponding to their field strength $F_{\mu\nu}=F^{+}_{\mu\nu}+F^{-}_{\mu\nu}$. Similarly the $s=2$ field appears as a 'Weyl tensor' $R_{(\alpha\beta\gamma\delta)}$ and its conjugate $R_{(\dot\alpha\dot\beta\dot\gamma\dot\delta)}$ transforming as the $(2,0)$ and $(0,2)$ representation of $SL(2,\mathbb{C})$, respectively. Furthermore the $s=\frac{3}{2}$ field appears in the $(\frac{3}{2},0)+(0,\frac{3}{2})$ representation of $SL(2,\mathbb{C})$. Even if one did not know that this supermultiplet provides a unitary representation of $N=8$ conformal superalgebra one could deduce, from the field content alone, that the corresponding supergravity must involve Weyl gravity. If there indeed exists a conformal $N=8$ supergravity theory whose fields are those of the $N=8$ conformal doubleton multiplet then we expect the 70 scalars to parametrise the coset space $E_{7(7)}/SU(8)$ as in the $N=8$ Poincaré supergravity. On the other hand parity invariant combinations of the fields in the $N=8$ conformal doubleton break the SU(8) symmetry down to SO(8). This suggests that the $N=8$ conformal supegravity may have a strong connection with the $N=8$ Poincaré supergravity.

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